A Graphical Method for Solution of Freezing Problems

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Problems in heat conduction involving a moving boundary are encountered in the freezing of liquids and in other situations. Such problems are difficult to solve, and exact solutions are almost unknown. A graphical method for obtaining numerical solutions to problems of this type which can be described in terms of one space coordinate is derived and is demonstrated in two examples involving the freezing of liquids. The method, which does not require specialized knowledge or equipment, takes into account both sensible heats and latent heat.

The rates of solidification of liquids have been of interest to engineers for a long time, the freezing problem being encountered in the manufacture of ice, in the casting of metals, in the preparation of cast explosive loadings in ordnance plants, in the processing of frozen foods, and in other circumstances. The mathematical problem of unsteady diffusion of heat in solids has received a large amount of attention, as evidenced by extensive treatments (*2*, *6*); however the special case of unsteady diffusion of heat involving a moving boundary, such as the problem of freezing, is intractable mathematically, and there is little published work to be found on this subject.

Neumann obtained a particular solution for freezing, or melting, in the semiinfinite region bounded by a plane. The solution, described by Carslaw and Jaeger (*2*) and Ingersoll et al. (*6*), neglects change of volume with freezing but takes into account differing thermal properties of the two phases. It applies for constant temperature initial and boundary conditions. Danckwerts (*4*) described solutions for a plane boundary similar to Neumann's but taking into account changes in volume. Lightfoot (*10*) treated solidification in the linear case by using the concept of moving sources. He assumed the thermal properties of the solid and of the liquid to be the same. Allen and Severn (*1*) used relaxation methods for numerical solution of the freezing problem in a semiinfinite medium with a plane boundary. Crank (*3*) recently gave methods for numerical solution of moving-boundary problems where the boundary is a plane. Pekeris and Slichter (*13*) described an approximate solution for ice formation on the outside of a cylinder when the liquid is initially at the freezing point. London and Seban (*11*) gave approximate solutions to the freezing problem for the slab, cylinder, and sphere. These solutions neglect the specific heats of the two phases, and in a later paper Seban and London (*16*) experimentally verified these solutions under conditions in which the neglect of specific heats was not serious. Fujita (*5*) described approximate analytic solutions for diffusion with a moving boundary for a semiinfinite medium and for the infinite cylinder for specific boundary conditions involving diffusion on only one side of the boundary. In the latter case the solution is apparently quite laborious. Keller and Ballard (*9*) treated the freezing of orange juice in cylindrical shape by using an effective thermometric conductivity which included the latent heat of fusion. They considered this treatment appropriate for orange juice since the juice does not have a definite freezing point.

EQUATIONS DESCRIBING FREEZING

Application of the principles of conservation of energy and of conservation of matter to a continuous phase having constant thermal conductivity and being held at constant pressure yields the following relationship, if kinetic and potential energies are neglected and if there are no distributed sources of energy:

$$K\nabla^2 t - u\cdot\nabla t = \frac{\partial t}{\partial \theta} \qquad (1)$$

Equation (1) includes the transport of energy by motion of the fluid.

If it is assumed that there is a definite interface between the solid and liquid phases and that the latent heat of fusion is liberated, or absorbed, at this interface, consideration of energy transfer at the interface leads to the equation

$$k_1(\nabla t_1)_i - k_2(\nabla t_2)_i = L\sigma_1 U \qquad (2)$$

If the phase change occurs at a definite temperature, the temperatures of both phases at the interface are described by

$$(t_1)_i = (t_2)_i = t_f \qquad (3)$$

If there is a change in volume accompanying the phase change, there is a hydrodynamic velocity of the liquid at the interface given by the equation

$$(u_2)_i = \left(1 - \frac{\sigma_1}{\sigma_2}\right)U \qquad (4)$$

A coordinate system fixed with respect to the solid phase is assumed.

If the specific weight of the liquid phase is assumed constant, the equation of continuity for the liquid phase reduces to the equation

$$\nabla\cdot u_2 = 0 \qquad (5)$$

The initial temperature distribution in each phase, the initial location of the interface, and the boundary conditions imposed must also be known in order to define the situation completely.

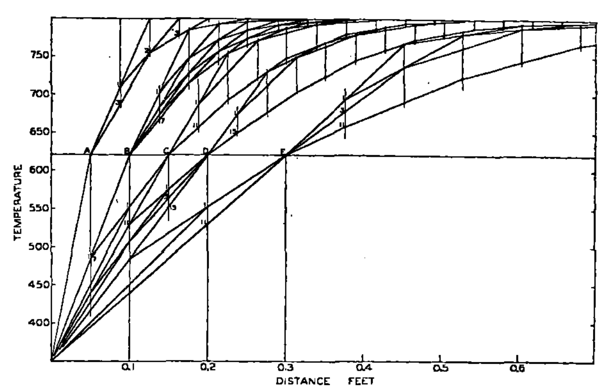

Fig. 1. Graphical construction for example 1.

DERIVATIONS FOR GRAPHICAL SOLUTION

The hydrodynamic velocity shown in Equation (1) and determined by Equations (4) and (5), and perhaps by certain boundary conditions, serves to complicate a problem which is already difficult. It is probably permissible, however, in the absence of forced convection to neglect hydrodynamic velocities, and they are so neglected in this paper. The magnitude of the error caused by this neglect can of course be estimated after solution of the simplified problem has been obtained.

With the neglect of hydrodynamic velocity, differential Equations (1) through (3) can be solved by graphical means when only one coordinate is involved. These graphical means are extensions of the method described by Schmidt (*14*, *15*) for the investigation of transient linear heat conduction in solids. The case of linear flow of heat is considered first. Equation (1) becomes for the solid and liquid phases respectively,

$$\frac{\partial t_1}{\partial \theta} = K_1 \frac{\partial^2 t_1}{\partial x^2} \quad 0 \leq x \leq X \qquad (6)$$

and

$$\frac{\partial t_2}{\partial \theta} = K_2 \frac{\partial^2 t_2}{\partial x^2} \quad X \leq x \qquad (7)$$

Equation (2) for the linear case becomes

$$k_1\left(\frac{\partial t_1}{\partial x}\right)_i - k_2\left(\frac{\partial t_2}{\partial x}\right)_i = L\sigma_1 \frac{dX}{d\theta} \qquad (8)$$

When the boundaries are stationary, Equations (6) and (7) are easily solved by the conventional Schmidt graphical construction. This construction requires that the intervals in time and distance be related by the equation

$$(\Delta x)^2 = 2K\Delta\theta \qquad (9)$$

and since the two phases have different thermometric conductivities, for equal time intervals the distance intervals for the two phases must be related by

$$\frac{\Delta x_1}{\Delta x_2} = \sqrt{\frac{K_1}{K_2}} \qquad (10)$$

However, one boundary for each of Equations (6) and (7) moves at a velocity determined by Equation (8). For this reason the solution to Equations (6), (7), and (8) is approximated by solving Equations (6) and (7) for a series of fixed boundaries which are changed at intervals controlled by Equation (8). Equation (8) may be written in finite-difference form as

$$k_1\left(\frac{\Delta t_1}{\Delta x_1}\right)_{i\,avg} - k_2\left(\frac{\Delta t_2}{\Delta x_2}\right)_{i\,avg} = L\sigma_1 \frac{\Delta X}{N\Delta\theta} \qquad (11)$$

in which $N\Delta\theta$ is the time necessary for the interface to move a distance ΔX. If the operation of averaging is included in Equation (11), it becomes

$$\sum_1^N \left[k_1\left(\frac{\Delta t_1}{\Delta x_1}\right)_i - k_2\left(\frac{\Delta t_2}{\Delta x_2}\right)_i\right] = L\sigma_1 \frac{\Delta X}{\Delta\theta} \qquad (12)$$

and since the increment in the interface position ΔX is taken equal to Δx_1, rearrangement of Equation (12) and use of Equations (9) and (10) lead to the relationship

$$\sum_1^N \left[\Delta t_1 - \frac{k_2}{k_1}\sqrt{\frac{K_1}{K_2}}\,\Delta t_2\right] = \frac{2L}{C_{p1}} \qquad (13)$$

which is the governing equation at the interface. The temperature increments in Equation (13) are measured over the interval next to the interface.

The time for the interface to move a distance ΔX is determined by moving the interface by a distance Δx_1 and then performing Schmidt constructions in each phase. The summation of the left side of Equation (13) is calculated after each time interval $\Delta\theta$, and N is determined by satisfaction of the equality.

Example 1 Linear Flow of Heat with Change of Phase

Since Neumann's solution (*2*, *6*) is an exact solution if the hydrodynamic velocity is ignored, a graphical solution was made of a problem for which Neumann's solution applied, and the results of the two methods were compared. Furthermore, the situation chosen is one in which the sensible heats of each phase are of the same order as the latent heat of fusion, since approximations neglecting either sensible heats or latent heat should be least applicable in this case. The problem selected may be stated as follows:

A large volume of molten lead is initially at a uniform temperature of 800°F. A plane at $x = 0$ is suddenly cooled to 350°F. and held at this temperature. Determine the position of the liquid-solid interface as a function of time and determine the temperature distribution at a selected time.

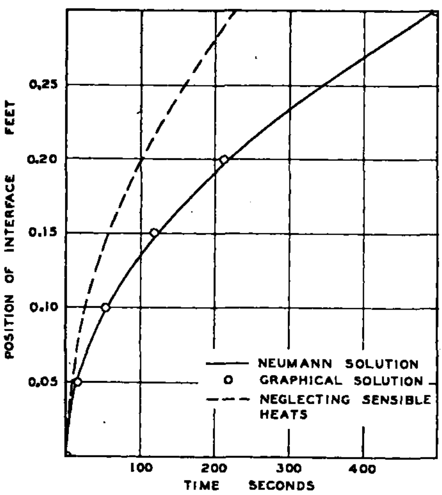

Fig. 2. Position of liquid-solid interface, example 1.

The situation is described mathematically by Equations (6), (7), and (8) and the initial and boundary conditions:

$$t = 800°F. \quad \text{at} \quad 0 \leq x \leq \infty \quad \text{for} \quad \theta < 0 \qquad (14)$$

$$t = 350°F. \quad \text{at} \quad x = 0 \quad \text{for} \quad \theta > 0 \qquad (15)$$

and

$$t_1 = t_2 = 621°F. \quad \text{at} \quad x = X \qquad (16)$$

The physical data used were obtained from the International Critical Tables (*7*) and temperature dependence of the properties was neglected. It was found that, of the total energy liberated in cooling lead from 800° to 350°F., approximately 23.7% is released in cooling the liquid to the freezing temperature of 621°F., 40.2% is released during freezing, and 36.1% is released in cooling the solid from 621° to 350°F.

Substituting numerical data in Equation (13) yields the following equation:

$$\sum_1^N (\Delta t_1 - 0.696\Delta t_2) = 601 \qquad (17)$$

To start a graphical solution Δx_1 was taken as 0.05 ft. By use of Equation (10) Δx_2 was found to be 0.038 ft. and Equation (9) gave $\Delta\theta$ as 5.69 sec. The construction is shown in Figure 1, and point A represents the position of the interface for the first calculation. Δt_1 was constant for this calculation at 271°F. and Δt_2 was determined by the graphical construction. Since temperature is determined graphically at the first interval from the interface only at odd values of N, the value of temperature to be used for even values of N must be arbitrarily decided. It is considered that use of the same temperature as determined for the previous odd value of N is appropriate here and that a better estimate of the average temperature gradient at the interface is obtained in this way than would be obtained by interpolation. Accordingly, for use in Equation (17) the temperatures at odd N were determined by the construction of Figure 1 and those at even N were taken as the same as at the preceding odd N. The numbers at points in Figure 1 correspond to N, and, in the case of point A, solution of Equation (17) gave $N = 2.83$, as shown in Table 1, which lists the solutions of Equation (17) for all intervals.

The position of the interface was then moved to point B of Figure 1 and graphical construction made as is shown in detail. The construction for subsequent positions of the interface is shown only in part because the lines came close together and would cause confusion in the small-scale reproduction. Constructions with the same increments in x were made for increments A through D, with the results shown in Table 1. Because the values of N necessary to solve Equation (17) were becoming large, the increment in x was doubled for point E, with the consequent quadrupling of the time interval $\Delta\theta$.

The detailed calculations for the solution at point C are shown in Table 2 to illustrate the method of determination of the values of N shown in Table 1.

The graphically determined position of the interface is shown in Figure 2 as a function of time, and the result of the analytical solution devised by Neumann (*2*) is also shown there. The agreement of the graphical solution with the analytical solution is good even though rather large intervals ΔX were used on the graphical solution. The dashed line represents the position of the interface calculated by neglecting the specific heats of the two phases (*11*). As would be expected for the chosen situation, large errors are introduced by neglect of the specific heats.

The distributions of temperature as determined by the graphical and by Neumann's analytical method were compared in Figure 3 for a time of 212 sec. after initiation of freezing. The graphical temperature distribution shown in Figure 3 was estimated by making arithmetic averages of the temperatures shown during the construction for point D of Figure 1, which corresponds to an elapsed time of 212 sec. The agreement of the temperatures determined by graphical means with those determined by Neumann's solution is close, in spite of the use in the graphical method of what appear to be rather crude approximations. The fact that the temperature gradients in the two phases at the interface appear to be the same in Figure 3 is fortuitous.

GRAPHICAL SOLUTION IN CYLINDRICAL SYMMETRY

In many applications involving freezing the geometry is such that the use of cylindrical coordinates is appropriate. If the phenomena considered are independent of angle and position along the axis, and if the hydrodynamic velocity is neglected, Equation (1) becomes the equations

$$K_1 \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial t_1}{\partial r}\right) = \frac{\partial t_1}{\partial \theta} \qquad (18)$$

$$r_0 \geq r \geq R$$

and

$$K_2 \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial t_2}{\partial r}\right) = \frac{\partial t_2}{\partial \theta} \quad r \leq R \qquad (19)$$

and Equation (2) may be expressed as

$$k_1\left(\frac{\partial t_1}{\partial r}\right)_i - k_2\left(\frac{\partial t_2}{\partial r}\right)_i = L\sigma_1 \frac{dR}{d\theta} \qquad (20)$$

Solution of Equations (18), (19), and (20) is possible if sufficient initial and boundary conditions are known.

Equations in cylindrical coordinates of the type shown in Equations (18) and (19) are conventionally (*8*, *15*) solved for situations where the boundaries are fixed by a Schmidt-type graphical construction in which the radius is divided into equal increments Δr which satisfy the relationship

$$(\Delta r)^2 = 2K\Delta\theta \qquad (21)$$

The graphical construction is performed on a plot having $\ln r$ as the abscissa, with the intervals in $\ln r$ selected to correspond with radii satisfying Equation (21), This approximation is satisfactory for cylindrical shells and is used here. A modification of this method is used for the solid phase near the center of the cylinder.

For definiteness the outside phase is taken as solid. This assumption corresponds to the case of the freezing of a cylinder having an external radius r_0. A change of variable is made in accordance with the relationship

$$w = -\ln\left(\frac{r}{r_0}\right) \qquad (22)$$

Equations (18) and (19) may be solved for a series of fixed boundaries by graphical construction by use of w as defined in Equation (22) as the abscissa, and with the increments in w selected by use of Equation (21). The time interval $\Delta\theta$ must be the same in both phases.

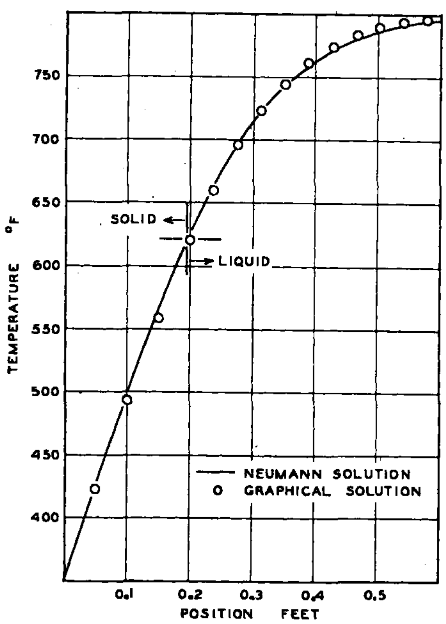

Fig. 3. Temperature distribution at 212 sec., example 1.

Use of Equation (22) in Equation (20) gives the equation

$$k_1\left(\frac{\partial t_1}{\partial w}\right)_i - k_2\left(\frac{\partial t_2}{\partial w}\right)_i = L\sigma_1 R^2 \frac{dW}{d\theta} \qquad (23)$$

when

$$W = -\ln\left(\frac{R}{r_0}\right) \qquad (24)$$

Equation (23) is written in finite difference form as

$$\sum_1^N \left[k_1\left(\frac{\Delta t_1}{\Delta w_1}\right)_i - k_2\left(\frac{\Delta t_2}{\Delta w_2}\right)_i\right] = L\sigma_1 R^2_{avg}\frac{\Delta W}{\Delta\theta} \qquad (25)$$

If the logarithmic mean radius (*12*) is used, the following equation applies:

$$R_{avg} = -\frac{\Delta R}{\Delta W} \qquad (26)$$

and substitution of Equation (26) in Equation (25) gives

$$\sum_1^N \left[k_1\left(\frac{\Delta t_1}{\Delta w_1}\right)_i - k_2\left(\frac{\Delta t_2}{\Delta w_2}\right)_i\right] = \frac{L\sigma_1(\Delta R)^2}{(\Delta W)(\Delta\theta)} \qquad (27)$$

as the equation to be satisfied at the interface between liquid and solid. Use of Equation (27) is similar to the use of Equation (13) for the linear case.

Example 2 Freezing in a Cylindrical System

The situation taken as an example of freezing in geometry having cylindrical symmetry is as follows.

Molten lead at an initially uniform temperature of 800°F. is contained in a cylindrical steel mold having an internal diameter of 4.0 in. and a wall thickness of ⅛ in. The mold is cooled externally by a fluid having a bulk temperature of 300°F. The heat transfer coefficient from this fluid to the mold may be taken as 300 B.t.u./(hr.)(sq. ft.)(°F.). Determine the position of the liquid-solid interface as a function of time after initiation of cooling and also determine the temperature distribution when the lead is almost totally frozen.

It was assumed that the height of the mold was large compared with the diameter, and so end effects could be neglected; hydrodynamic velocity was neglected also. The situation then is expressed mathematically by Equations (18), (19), and (20) and the initial conditions

$$t_2 = 800°\text{F.} \quad \text{at} \quad r_0 \geq r \geq 0 \quad \text{for} \quad \theta < 0 \qquad (28)$$

and

$$R = r_0 \quad \text{for} \quad \theta \leq 0 \qquad (29)$$

and the boundary conditions

$$\frac{\partial t}{\partial r} = 0 \quad \text{at} \quad r = 0 \qquad (30)$$

$$t_1 = t_2 = 621°\text{F.} \quad \text{for} \quad r = R \qquad (31)$$

and

$$\left(\frac{\partial t_1}{\partial r}\right)_{r=r_0} = \frac{t_s - t_0}{k_1\left[\frac{r_0}{hr_e} + \frac{r_0}{k_m}\ln\left(\frac{r_e}{r_0}\right)\right]} \qquad (32)$$

Equation (32) was derived by consideration of the external heat transfer. The specific heat of the steel mold was neglected, and the thermal flux from the lead was equated to that passing through the steel mold and into the coolant fluid. Use of Equation (22) in Equation (32) gives the equation

$$\left(\frac{\partial t_1}{\partial w}\right)_{w=0} = \frac{t_0 - t_s}{k_1\left[\frac{1}{hr_e} + \frac{1}{k_m}\ln\left(\frac{r_e}{r_0}\right)\right]} \quad (33)$$

A time interval $\Delta\theta$ was selected in accordance with the relationship

$$\Delta\theta = \frac{r_0^{\,2}}{128\mathrm{K}_1} \quad (34)$$

which corresponds to the time interval calculated from Equation (21) for eight equal divisions of radius in a solid cylinder. A time interval $\Delta\theta$ of 0.9864 sec. was calculated by use of Equation (34). Because the thermometric conductivities of the two phases are not equal, the radial intervals in the liquid phase must be recalculated for each new position of the liquid-solid interface by use of the formula

$$\Delta r_2 = \sqrt{\frac{\mathrm{K}_2}{\mathrm{K}_1}}\frac{r_0}{8} \quad (35)$$

The intervals Δw for the solid phase were calculated to correspond to intervals in the radius of one-eighth the internal radius of the mold.

The graphical construction is illustrated in Figure 4. Vertical lines at the values of w calculated for the solid phase are drawn in Figure 4 for temperatures less than the fusion temperature of 621°F. Vertical lines for temperatures greater than 621°F. are drawn as calculated by use of Equation (35) for the first four positions of the interface. Substitution of numerical data in Equation (33) gives as a boundary condition the equation

$$\left(\frac{\partial t}{\partial w}\right)_{w=0} = \frac{t_0 - 300}{0.3853} \quad (36)$$

This condition is satisfied by construction from a point located at 300°F. and a w of -0.3853, as shown in Figure 4.

Equation (27) may be rearranged to the form

$$C_1 + C_2\sum_1^N \Delta t_2 - \sum_1^N \Delta t_1 = 0 \quad (37)$$

The temperature increments used in Equation (37) are those for the increment in w closest to W, the interface position. The values of C_1 and C_2 in Equation (37) are shown in Table 3 for each step.

The graphical construction used in Figure 4 to solve Equation (37) is similar to that used for the linear semiinfinite case of Example 1 as shown in Figure 1. Only a portion of the construction lines is shown for each position of the interface in order to avoid confusion. The points from which values of Δt_1 and Δt_2 were read in Figure 4 are numbered to correspond to N in Equation (37), and values of Δt for even N were taken as the same as for the preceding odd N. The graphical construction for the liquid phase was done for the first four steps. Since the temperature of the liquid phase was quite close to the fusion temperature at the end of the fourth step, the sensible heat of the liquid was neglected for subsequent steps.

The construction was extended for one step beyond a radius ratio of 0.125 by methods which have not been published. It is possible to show that graphical solutions of the diffusion equation for solid cylinders are considerably improved in accuracy by taking the smaller radius of the innermost interval as equal to about 10 to 12% of the normal increment Δr rather than as zero. In the present case the innermost position was taken at a value of r/r_0 of 0.0133, corresponding to a w of 4.323. This position is not shown on Figure 4.

The values of N obtained and the times at which the interface was estimated to reach the indicated positions are shown in Table 3.

Figure 5 shows the position of the liquid-solid interface determined by this graphical solution as a function of time. The position calculated when the sensible heats of the two phases were neglected is also shown in Figure 5.

Temperature distributions for two positions of the liquid-solid interface are shown in Figure 6. The graphical temperatures were estimated by taking an arithmetic mean of the temperature indicated for each value of N without any interpolation between construction points. The dotted lines of Figure 6 show the logarithmic distribution which was obtained when sensible heats were neglected. The logarithmic distribution becomes a poor approximation to the temperature distribution in the solid phase as the liquid-solid interface approaches the center of the cylinder. The neglect of the sensible heat of the liquid phase implies that the liquid phase is always at the fusion temperature.

GRAPHICAL SOLUTION IN SPHERICAL SYMMETRY

Situations where the initial and boundary conditions possess spherical symmetry can be treated in spherical coordinates. If hydrodynamic velocities are neglected and temperature is a function only of the radial coordinate and time, Equation (1) has the form

$$\frac{\partial t}{\partial\theta} = \frac{\mathrm{K}}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) \quad (38)$$

and Equation (2) becomes

$$k_1\left(\frac{\partial t_1}{\partial r}\right)_i - k_2\left(\frac{\partial t_2}{\partial r}\right)_i = L\sigma_1\frac{dR}{d\theta} \quad (39)$$

Substitution of the variable w, defined by the equation

$$w = \frac{r_0}{r} \quad (40)$$

is made, and Equation (39) becomes

$$k_1\left(\frac{\partial t_1}{\partial w}\right)_i - k_2\left(\frac{\partial t_2}{\partial w}\right)_i = \frac{L\sigma_1 R^4}{r_0^{\,2}}\frac{dW}{d\theta}$$

$$= \frac{L\sigma_1 r_0^{\,2}}{W^4}\frac{dW}{d\theta} \quad (41)$$

Equations of the type of Equation (38) are conventionally solved for stationary boundaries by a Schmidt-type construction, with the variable w as abscissa and with the increments corresponding to equal increments in radius (*8, 15*). The graphical method can be extended to a series of fixed boundaries in the spherical case, as was done above for the cylindrical case. Equation (41), when expressed in finite difference form, becomes

$$\sum_1^N\left[k_1\left(\frac{\Delta t_1}{\Delta w_1}\right)_i - k_2\left(\frac{\Delta t_2}{\Delta w_2}\right)_i\right]$$

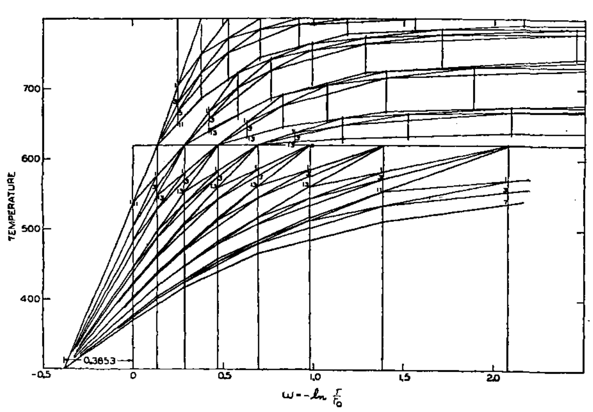

Fig. 4. Graphical construction for example 2.

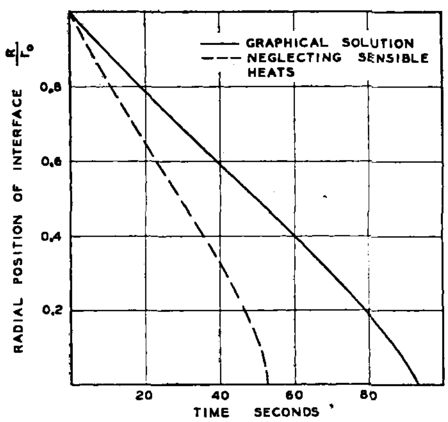

Fig. 5. Position of liquid-solid interface, example 2.

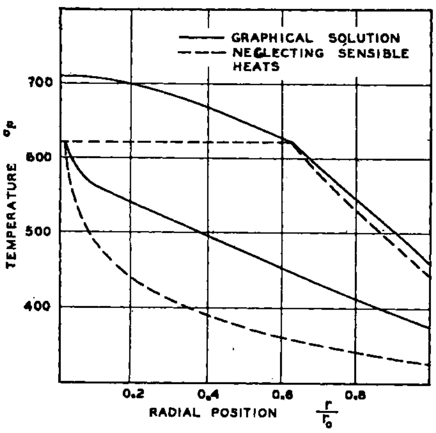

Fig. 6. Temperature distributions for two positions of interface, example 2.

TABLE 1. SOLUTIONS OF EQUATION (17) DETERMINED BY FIGURE 1

Interface position	X, ft.	Solution N	$\Delta\theta$, sec.	Time interval, sec.	Total elapsed time, sec.
A	0.05	2.82	5.69	16.0	16.0
B	0.10	6.63	5.69	37.7	53.7
C	0.15	11.47	5.69	65.3	119.0
D	0.20	16.35	5.69	93.0	212.0
E	0.30	12.46	22.76	283.6	495.6

TABLE 2. DETAILED SOLUTION OF EQUATION (17) FOR POINT C OF FIGURE 1

N	Δt_1, °F.	Δt_2, °F.	$0.696\Delta t_2$, °F.	Residual,* °F.
1	68	67	47	580
2	68	67	47	559
3	86	54	38	511
4	86	54	38	463
5	90	47	33	406
6	90	47	33	349
7	91	42	29	287
8	91	42	29	225
9	91	39	27	161
10	91	39	27	97
11	91	36	25	31
12	91	36	25	−35

Solution: $N = 11.47$

Time increment $= 11.47 \times 5.69 = 65.3$ sec.

*Residual $= 601 + 0.696 \sum_{1}^{N} \Delta t_2 - \sum_{1}^{N} \Delta t_1$.

TABLE 3. CONSTANTS FOR AND SOLUTIONS OF EQUATION (37)

Step	$\frac{R}{r_0}$	Constants C_1	Constants C_2	Solution N	Total elapsed time, sec.
1	0.875	601	0.613	11.53	11.37
2	0.750	601	0.598	12.62	23.79
3	0.625	601	0.589	13.21	36.81
4	0.500	601	0.563	12.75	49.36
5	0.375	601	—	13.29	62.46
6	0.250	601	—	12.15	74.44
7	0.125	601	—	11.21	85.49
8	0.0133	480	—	7.16	92.55

$$= \frac{L\sigma_1 R_{avg}^{\ 4} \Delta W}{r_0^{\ 2} \Delta\theta}$$

$$= \frac{L\sigma_1 r_0^{\ 2} \Delta W}{\Delta\theta W_A^{\ 2} W_B^{\ 2}} \qquad (42)$$

The geometric mean radius (*12*) is employed in Equation (42) since the geometry is spherical. Equation (42) is applied in the graphical solution for spherical symmetry in the manner that Equations (13) and (27) are used for linear and cylindrical cases respectively,

DISCUSSION

A procedure suitable for solution of moving-boundary problems such as are encountered with phase changes has been described and demonstrated. The method, which is graphical, does not require specialized equipment or difficult calculations and the time requirement is not large considering the relative complexity of the problem.

The derivations and the two examples chosen are for the freezing of a liquid; however the method of solution demonstrated should be suitable for other situations where the diffusion equation applies on each side of a moving boundary and the dependent variable has a constant value at the moving boundary.

While no mathematical proof is offered that a solution obtained by this method is indeed a good approximation to the exact solution, it is known that the Schmidt method of solution for fixed boundary problems is stable and does approach the exact solution as the size of the increments is reduced. It therefore appears that the graphical method demonstrated here should behave similarly, and certainly the results found in Example 1 agree very well with the exact solution even though the grid used was rather coarse.

NOTATION

C_1, C_2 = constants in Equation (37)
C_p = specific heat at constant pressure, B.t.u./(lb.)(°F.)
d = differential operator
h = heat transfer coefficient, B.t.u./(sq. ft.)(sec.)(°F.)
k = thermal conductivity, B.t.u./(ft.)(sec.)(°F.)
k_m = thermal conductivity of mold, B.t.u./(ft.)(sec.)(°F.)
L = latent heat of fusion, B.t.u./lb.
ln = natural logarithm
N = number of intervals
R = radius of liquid-solid interface, ft.
r = radius, ft.
r_0 = internal radius of mold, ft.
r_e = external radius of mold, ft.
t = temperature, °F.
t_f = fusion temperature, °F.
t_0 = temperature at internal surface of mold, °F.
t_s = bulk temperature of cooling medium, Example 2, °F.
U = vector velocity of liquid-solid interface, ft./sec.
u = vector hydrodynamic velocity, ft./sec.
W = value of w at liquid-solid interface, dimensionless
w = transformation variable, dimensionless
X = value of x at liquid-solid interface, ft.
x = Cartesian coordinate, ft.
Δ = increment in
θ = time, sec.
K = thermometric conductivity, sq. ft./sec.
$\sum$ = summation
σ = specific weight, lb./cu. ft.
∂ = partial differential operator
∇ = operator del of vector analysis
∇^2 = Laplacian operator of vector analysis
$\cdot$ = scalar product operator of vector analysis

Subscripts

1 = solid phase
2 = liquid phase
A, B = two sides of interval
avg = average
i = liquid-solid interface

LITERATURE CITED

1. Allen, D. N. De G., and R. T. Severn, *Quart. J. Mech. and App. Math.*, **5**, 447 (1952).
2. Carslaw, H. S., and J. C. Jaeger, "Conduction of Heat in Solids," Clarendon Press, Oxford (1947).
3. Crank, J., *Quart. J. Mech. and App. Math.*, **10**, 220 (1957).
4. Danckwerts, P. V., *Trans. Faraday Soc.*, **46**, 701 (1950).
5. Fujita, Hiroshi, *J. Chem. Phys.*, **21**, 700 (1953).
6. Ingersoll, L. R., Otto J. Zobel, and A. C. Ingersoll, "Heat Conduction," McGraw-Hill Book Company, Inc., New York (1948).
7. International Critical Tables, vols. 1 and 5, McGraw-Hill Book Company, Inc., New York (1926, 1929).
8. Jakob, Max, "Heat Transfer," vol. 1, John Wiley and Sons, Inc., New York (1949).
9. Keller, G. J., and J. H. Ballard, *Ind. Eng. Chem.*, **48**, 188 (1956).
10. Lightfoot, N. M. H., *Proc. London Math. Soc.* (2), **31**, 97 (1929).
11. London, A. L., and R. A. Seban, *Trans. Am. Soc. Mech. Engrs.*, **65**, 771 (1943).
12. McAdams, W. H., "Heat Transmission," 3 ed., McGraw-Hill Book Company, Inc., New York (1954).
13. Pekeris, C. L., and L. B. Slichter, *J. Appl. Phys.*, **10**, 135 (1939).
14. Schmidt, Ernst, "Festschrift zum Siebzigsten Geburtstag August Foeppls," p. 179, Julius Springer, Berlin (1924).
15. Schmidt, Ernst, *Forsch. Gebiete Ingenieurw.*, **13**, 177 (1942).
16. Seban, R. A., and A. L. London, *Trans. Am. Soc. Mech. Engrs.*, **67**, 39 (1945).

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